Faculty of Science, Technology, Engineering and Mathematics
MST210 Mathematical methods, models and modelling

Solutions to specimen exam paper

Solutions to Section 1

Summary of correct options

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Option	B	B	C	B	C	A	C	A	C	A	B	A	A	A	D	A	B	B	C

Solution to question 1

The characteristic equation of the given differential equation is

$$\lambda^2 + 2\lambda + 5 = 0.$$

This has solution

$$\lambda = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i.$$

So the solution of the given differential equation is

$$y = e^{-x} \left(A\cos(2x) + B\sin(2x) \right).$$

The correct option is **B**.

Solution to question 2

We are given the vectors $\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Let θ be the angle between the two vectors. Then we have

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos\theta.$$

So we calculate $\mathbf{a} \cdot \mathbf{b} = 1$, $|\mathbf{a}| = 1$ and $|\mathbf{b}| = \sqrt{1 + 1 + 1} = \sqrt{3}$. Substituting these into the above equation gives $1 = \sqrt{3}\cos\theta$, so

$$\theta = \arccos\left(\frac{1}{\sqrt{3}}\right).$$

The correct option is **B**.

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Solution to question 3

Differentiating the given vector gives

$$\dot{\mathbf{r}}(t) = -2\pi \sin(2\pi t)\mathbf{i} + 2\pi \cos(2\pi t)\mathbf{j} + \mathbf{k},$$

$$\ddot{\mathbf{r}}(t) = -4\pi^2 \cos(2\pi t)\mathbf{i} - 4\pi^2 \sin(2\pi t)\mathbf{j}.$$

So the magnitude of the acceleration is given by

$$|\ddot{\mathbf{r}}(t)| = \sqrt{(-4\pi^2 \cos(2\pi t))^2 + (-4\pi^2 \sin(2\pi t))^2} = 4\pi^2,$$

using the relation $\cos^2\theta + \sin^2\theta = 1$.

The correct option is **C**.

Solution to question 4

The first step is to make the lower-left entry zero. This is accomplished by replacing the second row with the result of adding twice the top row to the second row, that is,

$$\mathbf{R}_{2\mathrm{a}} = \mathbf{R}_2 + 2\mathbf{R}_1.$$

The correct option is **B**.

Solution to question 5

The eigenvalues of $\mathbf{A}$ are given as 1, 3 and 5.

Using rules on page 53 of the Handbook, the eigenvalues of $\mathbf{A} - 2\mathbf{I}$ are -1, 1 and 3.

Similarly, the eigenvalues of $(\mathbf{A} - 2\mathbf{I})^{-1}$ are -1, 1 and $\frac{1}{3}$.

The correct option is **C**.

Solution to question 6

Starting from the given general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{20t} + \beta \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-10t},$$

substitute $x = 1$, $y = -7$ and $t = 0$ to get

$$\begin{pmatrix} 1 \\ -7 \end{pmatrix} = \alpha \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

This gives the following two simultaneous linear equations to be solved:

$$1 = -3\alpha + \beta,$$

$$-7 = \alpha + 3\beta.$$

The solution of these equations is $\alpha = -1$ and $\beta = -2$.

Substituting these values into the general solution gives the required particular solution:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{20t} + \begin{pmatrix} -2 \\ -6 \end{pmatrix} e^{-10t}.$$

This question is arguably simpler to answer by looking at the options and seeing which option corresponds to the given initial conditions. Substituting $t = 0$ into the right-hand side of option A gives the initial position at $(3 + (-2) \quad -1 + (-6))^T = (1 \quad -7)^T$, which is the given initial

position, so this option is correct. If this were not correct, then we would try options B, C and D until the correct option was found.

The correct option is **A**.

Solution to question 7

Given

$$f(u, v) = \ln(u^2 + v^2) - uv,$$

differentiate to get

$$\frac{\partial f}{\partial u} = \frac{1}{u^2 + v^2} \times 2u - v.$$

The correct option is **C**.

Solution to question 8

The length of the spring is $l_0 - \frac{3}{4}l_0 = \frac{1}{4}l_0$.

Using Hooke's law for the force **H** due to the spring gives

$$\mathbf{H} = k\left(\tfrac{1}{4}l_0 - l_0\right)(-\mathbf{i}) = \tfrac{3}{4}kl_0\mathbf{i}.$$

The correct option is **A**.

Solution to question 9

The system will oscillate if it is weakly damped, and the condition for weak damping is $r^2 < 4mk$ (in the Handbook notation). In this case the stiffness is $3k$ and the damping constant is $2r$. So the system will oscillate if

$$(2r)^2 < 4 \times m \times 3k,$$

that is, if

$$r^2 < 3mk.$$

The correct option is **C**.

Solution to question 10

The normal mode angular frequencies are the square roots of the negatives of the eigenvalues of the dynamic matrix. In this case the dynamic matrix is upper triangular, so the eigenvalues can be read off the diagonal to be -1 and -4. So the normal mode angular frequencies are $\sqrt{-(-1)} = 1$ and $\sqrt{-(-4)} = 2$.

The correct option is **A**.

Solution to question 11

To classify the equilibrium point we need to find the eigenvalues of the matrix, so we need to find the roots of

$$\begin{vmatrix} -\lambda & -2 \\ 2 & 2 - \lambda \end{vmatrix} = 0.$$

This gives the characteristic equation $\lambda^2 - 2\lambda + 4 = 0$, which has roots

$$\lambda = \frac{2 \pm \sqrt{4 - 16}}{2} = 1 \pm \sqrt{3}i.$$

So the eigenvalues are complex with a positive real part.

Thus the equilibrium point is a spiral source.

The correct option is **B**.

Solution to question 12

The key words in this question are *could be* – this question is asking you not to calculate the Fourier series of the given function but merely to eliminate the options that could not be the Fourier series. This skill of finding errors by performing quick checks will be useful after you finish this module.

Here the given function is odd, so the Fourier series should contain only odd terms (this eliminates options C and D).

Options A and B differ only in the constant term. The constant term is given by the mean value of the function over the fundamental interval, which in this case is zero as the function is odd. So this eliminates option B.

This leaves option A as the only option that could be the Fourier series of the given function.

The correct option is **A**.

Solution to question 13

Substituting $u(x,t) = X(x)\,T(t)$ into the first boundary condition gives

$$X(0)\,T(t) = 0.$$

As this equation must hold for all $t \geq 0$, for non-trivial solutions we must have

$$X(0) = 0.$$

Substituting $u(x,t) = X(x)\,T(t)$ into the second boundary condition gives

$$X'(2\pi)\,T(t) = 0.$$

As this equation must hold for all $t \geq 0$, for non-trivial solutions we must have

$$X'(2\pi) = 0.$$

So the correct boundary conditions are

$$X(0) = 0, \quad X'(2\pi) = 0.$$

The correct option is **A**.

Solution to question 14

The $\mathbf{e}_\theta$-component of $\boldsymbol{\nabla} g$ is given by

$$\frac{1}{r}\frac{dg}{d\theta} = \frac{1}{r}\frac{d}{d\theta}(r\sin\phi\sin\theta) = \sin\phi\cos\theta.$$

The correct option is **A**.

Solution to question 15

Start by looking in the Handbook for the formula for divergence in spherical coordinates, namely

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, F_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}.$$

Here we have $F_r = r^3\sin\theta$, $F_\theta = 0$ and $F_\phi = 0$, so

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial(r^5\sin\theta)}{\partial r} = \frac{1}{r^2}\times 5r^4\sin\theta = 5r^2\sin\theta.$$

The correct option is **D**.

Solution to question 16

First find the points of intersection of the two curves, where

$$\tfrac{1}{2}x = x - x^2.$$

Thus

$$x^2 - \tfrac{1}{2}x = 0, \quad \text{or} \quad x(x - \tfrac{1}{2}) = 0.$$

The roots of this equation give the x-coordinates of the points of intersection as 0 and $\frac{1}{2}$. So the x-integral for the area should be from $x = 0$ to $x = \frac{1}{2}$.

To get the area between the two curves, the integrand should be 1 and the y-integral should be from the lower curve to the upper curve, that is, from $y = \frac{1}{2}x$ to $y = x(1-x)$.

So the integral to calculate the area is

$$\int_{x=0}^{x=1/2}\left(\int_{y=x/2}^{y=x(1-x)} 1\,dy\right)dx.$$

The correct option is **A**.

Solution to question 17

The total kinetic energy before the collision is $\frac{1}{2}mu^2 + \frac{1}{2}mu^2 = mu^2$.

The total kinetic energy after the collision is $\frac{1}{2}(2m)(u/\sqrt{2})^2 = \frac{1}{2}mu^2$.

So kinetic energy has been lost in the collision, that is, the collision is inelastic.

After the collision the two particles coelesce and so their relative velocity is the zero vector. As their relative velocity before the colision is a non-zero vector this must mean that the coefficient of restituton is 0.

The correct option is **B**.

Solution to question 18

Start by writing the two forces acting on the particle in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$:

$$\mathbf{T} = -|\mathbf{T}|\mathbf{e}_r,$$

$$\mathbf{W} = Mg\cos\theta\,\mathbf{e}_r - Mg\sin\theta\,\mathbf{e}_\theta.$$

Applying Newton's second law to the particle gives

$$M\ddot{\mathbf{r}} = \mathbf{T} + \mathbf{W} = -|\mathbf{T}|\mathbf{e}_r + Mg\cos\theta\,\mathbf{e}_r - Mg\sin\theta\,\mathbf{e}_\theta.$$

Resolving in the radial direction gives

$$-Ml\dot{\theta}^2 = -|\mathbf{T}| + Mg\cos\theta.$$

The correct option is **B**.

Solution to question 19

The moment of inertia of a solid cylinder about its axis of symmetry is given in the Handbook as $\frac{1}{2}MR^2$.

So by the parallel axes theorem, the moment of inertia about the given axis (which is a distance R from the axis of symmetry) is

$$I = \tfrac{1}{2}MR^2 + MR^2 = \tfrac{3}{2}MR^2.$$

The correct option is **C**.

On the next page we show a completed CME form with the correct answers marked. Also shown in the third column are examples of incorrect marking of answers that would be classed as invalid by the computer marker.

COMPUTER MARKED EXAMINATION FORM

08-10

PART 1

PERSONAL IDENTIFIER
(NOT Examination Number)

A1234567

ASSIGNMENT NUMBER
(as given on the question paper)

MST210 81

NAME and INITIALS

A STUDENT

IMPORTANT NOTES

1. Please check that all of this form has been completed including the coloured boxes in Part 1 with your Personal Identifier, Module and Assignment Number.
2. Only use an HB pencil to complete this form and press firmly.
3. Shown opposite are examples of the correct marks to use. Do not use any other types of mark.

PLEASE DO NOT PUNCH HOLES IN THIS FORM

Fold this form INWARDS along the dotted line

PERSONAL IDENTIFIER

A	0	0	0	0	0	0	0
B	1	1	1	1	1	1	1
C	2	2	2	2	2	2	2
D	3	3	3	3	3	3	3
E	4	4	4	4	4	4	4
F	5	5	5	5	5	5	5
G	6	6	6	6	6	6	6
H	7	7	7	7	7	7	7
K	8	8	8	8	8	8	8
L	9	9	9	9	9	9	9
M							X
N							
P							
R							
S							
T							
U							
V							
W							
X							
Y							

CORRECT MARK

Pencil across between dots —

CANCELLATION MARK

To cancel a mark pencil in the coloured part of the cell. Do not rub out ■

MODULE AND ASSIGNMENT NUMBER

A	A	A	A	0	0	0		0
B	B	B	B	1	1	1		1
C	C	C	C	2	2	2	2	2
D	D	D	D	3	3	3	3	3
E	E	E	E	4	4	4	4	4
F	F	F	F	5	5	5	5	5
G	G	G	G	6	6	6	6	6
H	H	H	H	7	7	7	7	7
K	K	K	K	8	8	8	8	8
L	L	L	L	9	9	9	9	9
M	M	M	M					
N	N	N	N					
P	P	P	P					
R	R	R	R					
S	S	S	S					
T	T	T	T					
U	U	U	U					
W	W	W	W					
X	X	X	X					
Y	Y	Y	Y					
Z	Z	Z	Z					

OFFICE USE ONLY

0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8
9	9

OFFICE USE ONLY

Examples of incorrect marks

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PART 2

	ANSWER		ANSWER		ANSWER
1	A B C D E F G H ? U	21	A B C D E F G H ? U	41	A B C D E F G H ? U
2	A B C D E F G H ? U	22	A B C D E F G H ? U	42	A B C D E F G H ? U
3	A B C D E F G H ? U	23	A B C D E F G H ? U	43	A B C D E F G H ? U
4	A B C D E F G H ? U	24	A B C D E F G H ? U	44	A B C D E F G H ? U
5	A B C D E F G H ? U	25	A B C D E F G H ? U	45	A B C D E F G H ? U
6	A B C D E F G H ? U	26	A B C D E F G H ? U	46	A B C D E F G H ? U
7	A B C D E F G H ? U	27	A B C D E F G H ? U	47	A B C D E F G H ? U
8	A B C D E F G H ? U	28	A B C D E F G H ? U	48	A B C D E F G H ? U
9	A B C D E F G H ? U	29	A B C D E F G H ? U	49	A B C D E F G H ? U
10	A B C D E F G H ? U	30	A B C D E F G H ? U	50	A B C D E F G H ? U
11	A B C D E F G H ? U	31	A B C D E F G H ? U	51	A B C D E F G H ? U
12	A B C D E F G H ? U	32	A B C D E F G H ? U	52	A B C D E F G H ? U
13	A B C D E F G H ? U	33	A B C D E F G H ? U	53	A B C D E F G H ? U
14	A B C D E F G H ? U	34	A B C D E F G H ? U	54	A B C D E F G H ? U
15	A B C D E F G H ? U	35	A B C D E F G H ? U	55	A B C D E F G H ? U
16	A B C D E F G H ? U	36	A B C D E F G H ? U	56	A B C D E F G H ? U
17	A B C D E F G H ? U	37	A B C D E F G H ? U	57	A B C D E F G H ? U
18	A B C D E F G H ? U	38	A B C D E F G H ? U	58	A B C D E F G H ? U
19	A B C D E F G H ? U	39	A B C D E F G H ? U	59	A B C D E F G H ? U
20	A B C D E F G H ? U	40	A B C D E F G H ? U	60	A B C D E F G H ? U

Solutions to Section 2

Solution to question 20

The given equation is

$$\frac{dy}{dx} = (y+1)\sin x.$$

This equation is separable, so we use the method of separation of variables to solve it. (The equation is also linear, so the integrating factor method could also be used after the equation is suitably rearranged. Here we use the separation of variables technique as it is quicker in this case.)

Separating the variables gives

$$\int \frac{dy}{y+1} = \int \sin x \, dx.$$

Integration yields

$$\ln(y+1) = -\cos x + C.$$

We are given that $y = 0$ when $x = 0$, so

$$\ln 1 = -\cos 0 + C,$$

which gives $C = 1$.

So the particular solution is

$$\ln(y+1) = 1 - \cos x.$$

Rearranging this gives

$$y + 1 = \exp(1 - \cos x),$$

so the required form for the solution is

$$y = \exp(1 - \cos x) - 1.$$

Solution to question 21

(a) To begin, calculate the first partial derivatives f_x and f_y of f:

$$f_x = 4x - 4y - 4, \quad f_y = -4x - 6y + 4.$$

At a stationary point $f_x(x, y) = f_y(x, y) = 0$, so

$$4x - 4y - 4 = 0,$$

$$-4x - 6y + 4 = 0.$$

Adding these equations gives $-10y = 0$, which gives $y = 0$ and hence $x = 1$.

So the stationary point is $(1, 0)$.

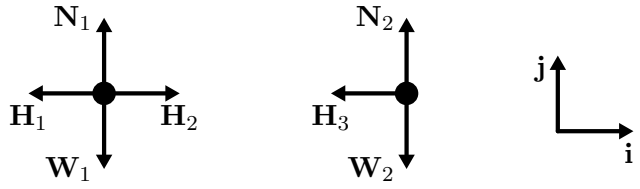
(b) In order to use the $AC - B^2$ test, we calculate the second partial derivatives of f:

$$A = f_{xx} = 4, \quad B = f_{xy} = -4, \quad C = f_{yy} = -6.$$

So $AC - B^2 = 4 \times (-6) - (-4) \times (-4) = -40$, which is negative. Thus the stationary point is a saddle.

Solution to question 22

Let $\mathbf{H}_1$, $\mathbf{H}_2$ be the forces exerted on particle B by the springs of stiffnesses $2k$ and k. Let $\mathbf{H}_3$ be the force exerted by the spring on particle C. Let $\mathbf{W}_1$, $\mathbf{W}_2$ be the two weights and $\mathbf{N}_1$, $\mathbf{N}_2$ be the two normal reaction forces, as shown in the force diagram below.



Now using the formula for the change in spring force from equilibrium, $\Delta\mathbf{H} = k\,\Delta l\,\widehat{\mathbf{s}}$, we have

$$\Delta\mathbf{H}_1 = 2kx(-\mathbf{i}),$$

$$\Delta\mathbf{H}_2 = k(y-x)\mathbf{i},$$

$$\Delta\mathbf{H}_3 = -\Delta\mathbf{H}_2 = k(x-y)\mathbf{i}.$$

Applying the formula for the change from equilibrium gives

$$m\ddot{x}\mathbf{i} = \Delta\mathbf{H}_1 + \Delta\mathbf{H}_2,$$

$$2m\ddot{y}\mathbf{i} = \Delta\mathbf{H}_3.$$

Resolving in the $\mathbf{i}$-direction gives

$$m\ddot{x} = -2kx + k(y-x) = -3kx + ky,$$

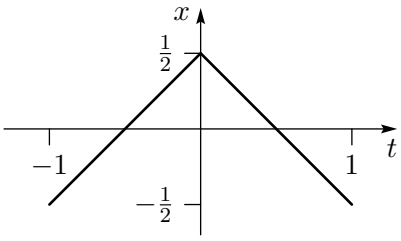
$$2m\ddot{y} = kx - ky.$$

These equations can be written in matrix form as

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \begin{pmatrix}-3k/m & k/m\\ \frac{1}{2}k/m & -\frac{1}{2}k/m\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

Solution to question 23

A sketch of the even extension is given below.

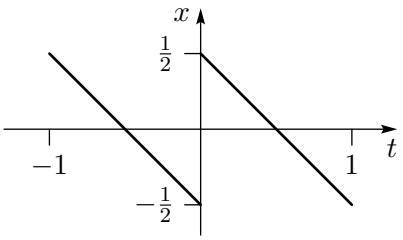


From this sketch we can write down the equations corresponding to the even extension:

$$f_{\text{even}}(t) = \begin{cases}\frac{1}{2} - t, & 0 \le t < 1,\\ \frac{1}{2} + t, & -1 \le t < 0,\end{cases}$$

$$f_{\text{even}}(t+2) = f_{\text{even}}(t).$$

A sketch of the odd extension is given below.



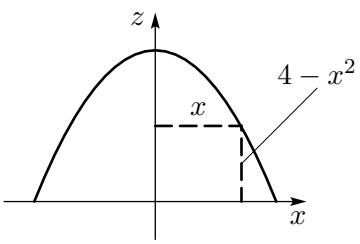
From this sketch we can write down the equations corresponding to the odd extension:

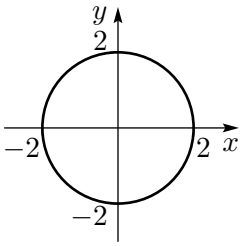
$$f_{\text{odd}}(t) = \begin{cases} \frac{1}{2} - t, & 0 < t < 1, \\ -\frac{1}{2} - t, & -1 < t < 0, \\ 0 & t = -1, 0, 1, \end{cases}$$

$$f_{\text{odd}}(t+2) = f_{\text{odd}}(t).$$

Solution to question 24

The first step is to draw side and plan views of the region of integration, as shown below.





As the solid is axially symmetric, we use cylindrical coordinates.

From the side view, the z-range at a distance x from the origin is from 0 to $4 - x^2$.

From the plan view, the ρ-range is from 0 to 2.

From the Handbook, the volume integral for a function that is independent of ϕ in cylindrical coordinates is

$$V = 2\pi \int_{\rho=0}^{\rho=2} \left(\int_{z=0}^{z=4-\rho^2} \rho \, dz \right) d\rho.$$

Evaluating the inner integral gives

$$V = 2\pi \int_{\rho=0}^{\rho=2} \left[\rho z\right]_{z=0}^{z=4-\rho^2} d\rho = 2\pi \int_{\rho=0}^{\rho=2} (4\rho - \rho^3)\, d\rho.$$

Now evaluate the ρ-integral:

$$V = 2\pi \left[2\rho^2 - \tfrac{1}{4}\rho^4\right]_{\rho=0}^{\rho=2} = 2\pi(8-4) = 8\pi.$$

Solution to question 25

(a) Applying the principle of conservation of momentum to the collision gives

$$mu\mathbf{i} + mu\mathbf{j} = 2m(v\mathbf{i} + w\mathbf{j}).$$

Resolving into **i**- and **j**-components, and dividing by $2m$, gives

$$v = \tfrac{1}{2}u, \quad w = \tfrac{1}{2}u.$$

So the speed after the collision is given by

$$\sqrt{v^2 + w^2} = \sqrt{\left(\tfrac{1}{2}u\right)^2 + \left(\tfrac{1}{2}u\right)^2} = \tfrac{1}{\sqrt{2}}u.$$

(b) The kinetic energy before the collision is

$$\tfrac{1}{2} \times m \times u^2 + \tfrac{1}{2} \times m \times u^2 = mu^2.$$

The kinetic energy after the collision is

$$\tfrac{1}{2} \times 2m \times \left(\tfrac{1}{\sqrt{2}}u\right)^2 = \tfrac{1}{2}mu^2.$$

So the ratio of kinetic energies is

$$\frac{\frac{1}{2}mu^2}{mu^2} = \frac{1}{2}.$$

Solutions to Section 3

Solution to question 26

We are given

$$\frac{dx}{dt} = -4x + 6y - 3e^{-t}, \quad \frac{dy}{dt} = -9x + 11y + 3e^{-t}.$$

In matrix form this is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3e^{-t} \\ 3e^{-t} \end{pmatrix},$$

with characteristic equation $(-4-\lambda)(11-\lambda)+54 = 0$, or $\lambda^2 - 7\lambda + 10 = 0$.

This equation factorises as $(\lambda - 2)(\lambda - 5) = 0$, so the eigenvalues are $\lambda = 2$ and $\lambda = 5$.

For the eigenvalue $\lambda = 2$, the corresponding eigenvector satisfies

$$\begin{pmatrix} -6 & 6 \\ -9 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So $-6x + 6y = 0$. Hence $y = x$ and $(1 \quad 1)^T$ is an eigenvector.

For the eigenvalue $\lambda = 5$, the corresponding eigenvector satisfies

$$\begin{pmatrix} -9 & 6 \\ -9 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So $-9x + 6y = 0$. Hence $2y = 3x$ and $(2 \quad 3)^T$ is an eigenvector.

So the complementary function is

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{5t} \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

where A and B are constants.

To find the particular integral, since -1 is not an eigenvalue, consider the trial solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} pe^{-t} \\ qe^{-t} \end{pmatrix},$$

where p and q are constants whose values are to be determined.

Substituting into the matrix form of the given equations yields

$$\begin{pmatrix} -pe^{-t} \\ -qe^{-t} \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & 11 \end{pmatrix} \begin{pmatrix} pe^{-t} \\ qe^{-t} \end{pmatrix} + \begin{pmatrix} -3e^{-t} \\ 3e^{-t} \end{pmatrix}.$$

This holds for all t, so the coefficients of e^{-t} must be equal. Thus

$$-p = -4p + 6q - 3,$$

$$-q = -9p + 11q + 3.$$

These equations simplify to

$$3p - 6q = -3,$$

$$9p - 12q = 3.$$

Subtracting twice the first equation from the second equation gives $3p = 9$, so $p = 3$. Substituting this into the first equation gives $9 - 6q = -3$, so $q = 2$.

So a particular integral is $(x \quad y)^T = (3e^{-t} \quad 2e^{-t})^T$, and the general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{5t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + e^{-t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Using the initial conditions $x(0) = 0$ and $y(0) = 0$ gives

$$A + 2B + 3 = 0,$$

$$A + 3B + 2 = 0.$$

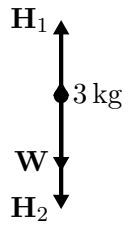
Subtracting the first equation from the second equation gives $B - 1 = 0$, so $B = 1$. Substituting this into the first equation gives $A + 2 + 3 = 0$, so $A = -5$.

So the particular solution of the system of simultaneous linear equations is

$$\begin{pmatrix} x \\ y \end{pmatrix} = -5e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{5t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + e^{-t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution to question 27

(a) The forces acting on the particle are the weight $\mathbf{W}$, the force $\mathbf{H}_1$ due to the spring AP, and the force $\mathbf{H}_2$ due to the spring PB. The force diagram is shown below.



(b) The weight is $\mathbf{W} = 3g\mathbf{i}$, and the spring forces are shown in the table below.

Spring	Force	Length	l_0	Extension	Stiffness	$\widehat{\mathbf{s}}$	$\mathbf{H}$
AP	$\mathbf{H}_1$	x	1	$x - 1$	18	$-\mathbf{i}$	$18(x-1)(-\mathbf{i})$
BP	$\mathbf{H}_2$	$\frac{3}{2} - x$	2	$\frac{3}{2} - x - 2$	30	$\mathbf{i}$	$30(-\frac{1}{2} - x)\mathbf{i}$

(c) Let x_{eq} be the particle's position when in equilibrium. In equilibrium we have

$$\mathbf{W} + \mathbf{H}_1 + \mathbf{H}_2 = \mathbf{0}.$$

Resolving in the $\mathbf{i}$-direction gives

$$3g - 18(x_{\text{eq}} - 1) + 30(-\tfrac{1}{2} - x_{\text{eq}}) = 0.$$

Solving for x_{eq} gives the equilibrium position $x_{\text{eq}} = (1 + g)/16$.

(d) The differential equation of motion is

$$\begin{aligned} 3\ddot{x}\mathbf{i} &= \mathbf{W} + \mathbf{H}_1 + \mathbf{H}_2 \\ &= 3g\mathbf{i} - 18(x-1)\mathbf{i} + 30(-\tfrac{1}{2} - x)\mathbf{i}. \end{aligned}$$

Resolving in the $\mathbf{i}$-direction and simplifying gives

$$\ddot{x} + 16x = 1 + g.$$

(e) The solution of the differential equation is

$$x = A\cos(4t) + B\sin(4t) + x_{\text{eq}}.$$

Differentiate this in order to use the initial conditions:

$$\dot{x} = -4A\sin(4t) + 4B\cos(4t).$$

The initial conditions are $x(0) = x_{\text{eq}} - \frac{1}{4}$, $\dot{x}(0) = 0$, and on substitution these give

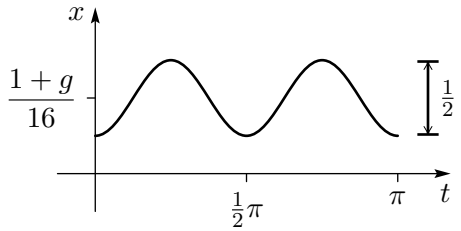
$$x_{\text{eq}} - \tfrac{1}{4} = A + x_{\text{eq}}, \quad 0 = 4B.$$

These equations have solution $B = 0$ and $A = -\frac{1}{4}$.

So the particular solution is

$$x = -\tfrac{1}{4}\cos(4t) + x_{\text{eq}}.$$

(f) The period is $2\pi/4 = \pi/2$ and the amplitude is $|-\frac{1}{4}| = \frac{1}{4}$. A sketch of x against t is shown below.



Solution to question 28

(a) Let $u(x,t) = X(x)\,T(t)$. Then

$$\frac{\partial^2 u}{\partial x^2} = X''T, \quad \frac{\partial u}{\partial t} = XT'.$$

Substituting into the differential equation and rearranging gives

$$\frac{X''}{X} = \frac{1}{\alpha}\frac{T'}{T}.$$

Since the left-hand side is a function of x alone, while the right-hand side is a function of t alone, the two expressions can be equal for all x and all t only if each is equal to a constant, say μ. Then

$$X'' - \mu X = 0, \quad T' - \mu\alpha T = 0.$$

The boundary conditions give $X(0) = X(\pi) = 0$.

We are given that μ is negative, so let $\mu = -k^2$ for some positive constant k. The above equation then becomes $X'' + k^2 X = 0$, which has general solution

$$X(x) = A\cos(kx) + B\sin(kx).$$

The boundary condition $X(0) = 0$ gives $A = 0$. The boundary condition $X(\pi) = 0$ gives either $B = 0$ or $\sin(k\pi) = 0$. If $B = 0$, then we have the trivial solution $X(x) = 0$. If $\sin(k\pi) = 0$, then k is an integer, n say.

So non-trivial solutions are

$$X(x) = B\sin(nx), \quad n = 1, 2, 3, \ldots,$$

where B is an arbitrary constant.

Since $\mu = -n^2$, the differential equation for T becomes

$$T' + \alpha n^2 T = 0,$$

which has general solution

$$T(t) = Ce^{-\alpha n^2 t}.$$

So the following function satisfies the differential equation plus boundary conditions:

$$u(x,t) = Be^{-\alpha n^2 t}\sin(nx), \quad n = 1, 2, 3, \ldots.$$

Thus by the principle of superposition, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\alpha n^2 t}\sin(nx),$$

where the B_n are constants.

(b) We must find the coefficients B_n such that

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin(nx) = f(x).$$

Hence we need to express $f(x)$ as a Fourier sine series, whose coefficients are given by

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx.$$

Solution to question 29

(a) Start by computing the curl of $\mathbf{F}$:

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz^2+y & -yz^2+x & z(x^2-y^2) \end{vmatrix} \\ &= (-2yz-(-2yz))\,\mathbf{i} - (2xz-2xz)\,\mathbf{j} + (1-1)\,\mathbf{k} \\ &= \mathbf{0}.\end{aligned}$$

The curl of $\mathbf{F}$ is zero, so by the curl test $\mathbf{F}$ is conservative.

(b) The given path is

$$\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},$$

so

$$\frac{d\mathbf{r}}{dt} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}.$$

Along this path we have

$$\begin{aligned}\mathbf{G}(\mathbf{r}(t)) = &\big((\cos t)\times 0^2 - \sin t\big)\,\mathbf{i} + \big((\sin t)\times 0^2 + \cos t\big)\,\mathbf{j} \\ &+ \big(0\times(\cos^2 t - \sin^2 t)\big)\,\mathbf{k} \\ = &-\sin t\,\mathbf{i} + \cos t\,\mathbf{j}.\end{aligned}$$

Thus

$$\mathbf{G}\cdot\frac{d\mathbf{r}}{dt} = (-\sin t)(-\sin t) + (\cos t)(\cos t) = 1.$$

So the line integral is

$$\int_C \mathbf{G}\cdot d\mathbf{r} = \int_0^{2\pi} 1\,dt = 2\pi.$$

Hence $\mathbf{G}$ is not the gradient of a scalar field, as the integral around any closed path is zero for any vector field that is the gradient of a scalar field.

(c) The given path is

$$\mathbf{r} = at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k},$$

so

$$\frac{d\mathbf{r}}{dt} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

Along this path we have

$$\mathbf{F}(\mathbf{r}(t)) = \left(ac^2t^3 + bt\right)\mathbf{i} + \left(-bc^2t^3 + at\right)\mathbf{j} + \left(c(a^2 - b^2)t^3\right)\mathbf{k}.$$

Thus

$$\begin{aligned}\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= a^2c^2t^3 + abt - b^2c^2t^3 + abt + c^2(a^2 - b^2)t^3 \\ &= 2c^2(a^2 - b^2)t^3 + 2abt.\end{aligned}$$

So the line integral is

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \left(2c^2(a^2 - b^2)t^3 + 2abt\right) dt \\ &= \left[\tfrac{1}{2}c^2(a^2 - b^2)t^4 + abt^2\right]_0^1 \\ &= \tfrac{1}{2}c^2(a^2 - b^2) + ab.\end{aligned}$$

Hence

$$U(a, b, c) = -\int_C \mathbf{F} \cdot d\mathbf{r} = -\left(\tfrac{1}{2}c^2(a^2 - b^2) + ab\right).$$

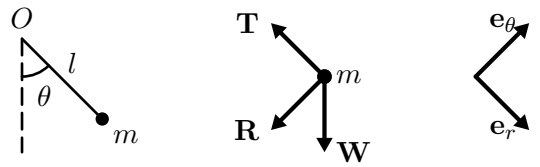
Because a, b and c are arbitrary, we can replace them with x, y and z to obtain

$$\begin{aligned}U(x, y, z) &= -\tfrac{1}{2}z^2(x^2 - y^2) - xy \\ &= \tfrac{1}{2}z^2(y^2 - x^2) - xy.\end{aligned}$$

Then we have $\mathbf{F} = -\boldsymbol{\nabla} U$ and $U(0, 0, 0) = 0$, as required.

Solution to question 30

(a) A picture of the system and a force diagram are shown below. The three forces are the weight $\mathbf{W}$, the tension force due to the rod $\mathbf{T}$, and the resistance force $\mathbf{R}$.



(b) The position vector of the pendulum bob with respect to O is $l\mathbf{e}_r$.

Write the three forces in terms of the unit vectors:

$$\mathbf{T} = -|\mathbf{T}|\mathbf{e}_r,$$

$$\mathbf{R} = -|\mathbf{R}|\mathbf{e}_\theta,$$

$$\mathbf{W} = mg\cos\theta\,\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta.$$

We are given that $|\mathbf{R}| = kv$, but the speed v is given by $v = l\dot{\theta}$, so

$$\mathbf{R} = -kl\dot{\theta}\,\mathbf{e}_\theta.$$

(c) We calculate the torque about O for each force in turn. All three forces are applied to the pendulum bob, which has position $l\mathbf{e}_r$, so

$$\boldsymbol{\Gamma}_T = (l\mathbf{e}_r) \times (-|\mathbf{T}|\mathbf{e}_r) = \mathbf{0},$$

$$\boldsymbol{\Gamma}_R = (l\mathbf{e}_r) \times (-kl\dot{\theta}\mathbf{e}_\theta) = -kl^2\dot{\theta}\mathbf{k},$$

$$\boldsymbol{\Gamma}_W = (l\mathbf{e}_r) \times (mg\cos\theta\,\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta) = -mgl\sin\theta\,\mathbf{k}.$$

So the total torque about O is

$$\boldsymbol{\Gamma} = -kl^2\dot{\theta}\mathbf{k} - mgl\sin\theta\,\mathbf{k}.$$

(d) By definition, angular momentum is $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{r}$ is the position of the pendulum bob and $\mathbf{p}$ is the linear momentum of the bob. The velocity of the bob is $l\dot{\theta}$, so $\mathbf{p} = ml\dot{\theta}\mathbf{e}_\theta$.

So the angular momentum about O is given by

$$\mathbf{L} = l\mathbf{e}_r \times ml\dot{\theta}\mathbf{e}_\theta = ml^2\dot{\theta}\mathbf{k}.$$

(e) By the torque law, $\dot{\mathbf{L}} = \boldsymbol{\Gamma}$, so

$$\frac{d}{dt}(ml^2\dot{\theta}\mathbf{k}) = -kl^2\dot{\theta}\mathbf{k} - mgl\sin\theta\,\mathbf{k}.$$

Resolving in the $\mathbf{k}$-direction and simplifying gives the required equation of motion:

$$\ddot{\theta} + \frac{k}{m}\dot{\theta} + \frac{g}{l}\sin\theta = 0.$$